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We consider the time development of the density matrix for a system coupled to a thermal bath, in models that go beyond the standard two-level systems through addition of an energy excitation degree of freedom and through the possibility of the replacement of the spin algebra by a more complex algebra. We find conditions under which increasing the coupling to the bath above a certain level decreases the rate of entropy production, and in which the limiting behavior is a dissipationless sinusoidal oscillation that could be interpreted as the synchronization of many modes of the uncoupled system.

Introduction. We explore some extensions of the problem of a two level system in a thermal bath [1] and find some new qualitative behavior that can be provided by these extensions. Two specific examples are the symmetrical double well in an environment with temperature on the order of the (single-well) level spacing, and the propagation of oscillating neutrinos in hot, dense media. In both cases we find behavior in which a weak coupling of the system to the bath induces some rate of entropy gain, but where an increase in the coupling beyond a certain value reduces the rate of entropy production, reducing it to zero in a limiting case. The limiting behavior can correspond to the "frozen" conditions of ref. [2,3], and many other papers, in which the implicit continuous measurement of a quantity by the system-bath interaction can be said to have frozen that quantity in time. However our main focus in the present note is on circumstances that give instead a kind of dissipationless collective oscillation, which gives the appearance of a synchronization of modes that had different frequencies in the absence of the coupling to the bath. We stress, however, that there appears to be little direct connection of these results to the large literature of synchronized behavior in non-linear systems [4–6] or of stochastic resonances in forced systems [7].

The system itself has two coordinates, which we characterize as "horizontal" and "vertical". The horizontal component is the "spin" coordinate of a two state system, or in cases that may be of interest in neutrino physics, a "flavor" coordinate for a 3 or 4-state system. We index these states with Greek letters. The vertical variable corresponds to a set of energy levels, E_i. The system basis states are denoted $|E_i, \alpha\rangle$. The dynamics of the system uncoupled to the bath is governed by the Hamiltonian,

$$H_0^{\rm sys} = \sum_{i,\alpha,\beta} \{E_i \delta_{\alpha,\beta} + \lambda_{\alpha,\beta}(E_i)\} \Big[|E_i, \alpha >< E_i, \beta| \Big]. \quad (1)$$

We take $\lambda_{\alpha,\beta}(E_i) \ll E_k$ for all values i, k. We consider a density matrix for the system $\rho^{\rm sys}(t) = {\rm Tr}_{\rm bath}[\rho^{\rm s+b}(t)]$, where $\rho^{\rm s+b}(t)$ is the complete density matrix of the model. At an initial time t=0 we choose a form diagonal in the vertical indices,

$$\rho^{\rm sys}(t=0) = \sum_{i,\alpha,\beta} \rho_{\alpha,\beta}(E_i, t=0) |E_i, \alpha\rangle\langle E_i, \beta|. \quad (2)$$

For these initial conditions, one of the results of a master equation derivation is that for times $t \gg (\Delta E)^{-1}$, where ΔE is a measure of the (single-well) energy spacings, the operator ρ^{sys} remains so nearly diagonal in the vertical (energy) indices that we can continue to describe the system by a vector in the energy space, $\rho(E_i, t)_{\alpha,\beta}$, replacing $(t = 0)$ by (t) in (2). Henceforth we suppress the indices (α, β) in the horizontal space, in which $\rho(E_i, t)$ remains an operator. For the bath-system coupling, H_{s-b}, we take the dependence on the horizontal (spin-flavor variables) to be in a factorized form, $H_{s-b} = \zeta V$, where ζ operates on the horizontal indices and V depends only on the vertical coordinates and the coordinates of the bath. This coupling is also taken to be weak, in a way that allows the problem to be solved, although it can at the same time be strong in another sense, for the realization of the limits described above. At the end of this note we show that for this form of coupling the Bloch equations, generalized to include the vertical structure, are of the form,

$$\frac{d}{dt}\rho(E_i, t) = -i[\lambda(E_i), \rho(E_i, t)] + \sum_{E_j} \zeta\, \rho(E_j, t)\, \zeta\, \Gamma(E_j, E_i) - \frac{1}{2}\Big(\zeta^2\, \rho(E_i, t) + \rho(E_i, t)\, \zeta^2\Big) \sum_{E_j} \Gamma(E_i, E_j). \quad (3)$$

All of the elements in (3) are matrices in the horizontal space except for the functions Γ. These functions obey the relation $\Gamma(E_j, E_i) = \exp[(E_j - E_i)/T]\Gamma(E_i, E_j)$ in consequence of the thermal equilibrium of the bath. For the neutrino application, and with the right identifications, (3) is the "Quantum Kinetic Equation" derived by McKellar and Thomson [8]. Also Loreti and Balantekin

[9] have used a special case of (3) to discuss the energy conserving case of propagation of neutrinos through a medium with a (spatially) fluctuating electron density, in which case the collision term becomes a pure double commutator in the horizontal (flavor) space. For the double well application, we sketch at the end of this note a derivation in the context needed for our present results.

A. Symmetrical double well

The horizontal space is 2x2, with $\sigma_3 = 1$ for the particle to be on the left side and $\sigma_3 = -1$ for the right side. The barrier height is is large compared to ΔE where ΔE is a measure of the energy spacing for the single well. The vertical states are the single well energy excitations. We choose $\lambda_{\alpha,\beta}(E_i) = [\sigma_1]_{\alpha,\beta} g(E_i)$. The energy splittings $g(E_i)$, which are essentially tunneling rates, have strong E_i dependence. For the "spin" dependence of the coupling to the bath we take, $\zeta = 1 + b\sigma_3$. In the case $b = 0$ the transition matrix elements of the operator V for $E_i \to E_j$ in the left-hand well, considered by itself, are the same as those for the right-hand well, as would be the case in the dipole approximation of the interaction with a radiation field. For the case $b = 1$ only the amplitude on the left-hand side of the well interacts with the bath.

The coefficient $\Gamma(E_i, E_j)$ is the (single well) rate for a state E_i to make a transition to state E_j. When $T \approx \Delta E$, where ΔE is the scale of the vertical splittings, the bath induces transitions among the vertical states, and in this case our models differ substantively from models that do not have the vertical structure.

B. Neutrinos

The question is the time dependence of the flavor density matrix for neutrinos that oscillate and are also scattered as they traverse a dense medium, like that of the supernova core, or the early universe at a temperature greater than 1 MeV. Here the $\lambda_{\alpha,\beta}$ parameters are the combination of the mass$^2/(2E)$ matrix and the "index of refraction" terms, the latter energy independent and of order G_W. The functions, $\Gamma(E_i, E_j)$, of order G_W^2, are the rates of transition for a neutrino to go from state E_i to state E_j, as calculated with the matrix ζ set equal to unity. The choice $b = 0$ would apply to ν_μ, ν_τ mixing in a flavor-blind medium, that is, a medium in which ν_μ and ν_τ have the same interactions with the environment. The choice $b = 1$ corresponds to "active-sterile" scenarios in which one of the neutrinos does not interact with the environment at all.

Results for $b = 0$. As an example we use a double square well with infinite barriers to the left and right of $x = \pm 8a$, and with a central barrier of width $2a$ and height U_0. We choose U_0 and the particle mass such that twenty states are bound with energies less than U_0. The low-lying states are thus very nearly the symmetrical and anti-symmetrical combinations of separate states in an infinite well of width $7a$. The energy splittings, $g(E_i)$ are easy to calculate in the limit of small tunneling. For the bath we take bosons with $\omega = vk$, equal energy spacing $2\pi v/L$ where $L \to \infty$, and with creation and annihilation operators, $a_n^\dagger, a_n$. We take $H_0 = \sum_n a_n^\dagger a_n \omega_n + H_0^{(\mathrm{sys})}$, and in the system-bath interaction, $H_{s-b} = V\zeta$, we take the vertical operator V to have off-diagonal matrix elements given in dipole expansion as *,

$$V_{j,k} = \frac{q}{L} \sum_n [(a_n^\dagger + a_n)\delta_{j,k} + ik_n x_{j,k}(a_n^\dagger - a_n)]. \qquad (4)$$

where q is a coupling strength. We calculate the off-diagonal dipole matrix elements, $x_{j,k}$ using the infinite well wave-functions and define $c_{j,k} = qk_n x_{j,k}$, where the later formalism will justify using (near) energy conservation to express k_n as $\pm(E_j - E_k)/v$. The coefficients in the bath interaction part of (3) are then,

$$\Gamma(E_j, E_k) = 2\pi c_{j,k} c_{k,j} \Big[\frac{\theta(E_j - E_k)}{e^{(E_j - E_k)/T} - 1} + \frac{\theta(E_k - E_j)}{1 - e^{(E_j - E_k)/T}} \Big]. \qquad (5)$$

Incorporating these results and solving (3) for several values of q, we show in Fig.1 the probability that the particle will be on the left side at time t, with the initial condition of being on the left and in a thermal distribution at the bath temperature, which we take as $T = 5$ in units of the single-well ground state energy.

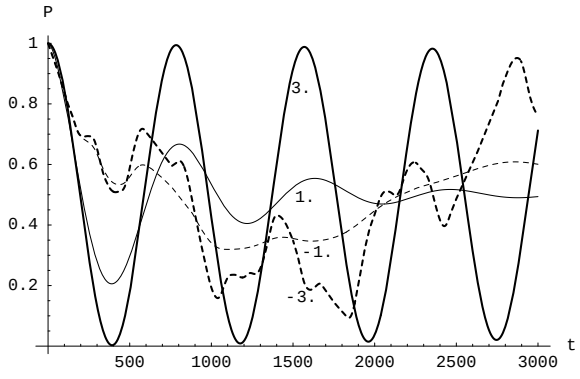

FIG. 1. The probability, P, for a particle to be found in the left hand well, for a bath temperature $T = 5$, in units of the ground state energy of the single infinite well. The initial condition is that the particle is on the left with a thermal distribution of energies. The integers (-3,-1,1,3) labeling the curves are the Log_{10} of the bath-system coupling strengths in arbitrary units.

*There are diagonal matrix elements of the kx operator as well, since the wells are not centered at $x = 0$. In principle, diagonal matrix elements contribute to (3) in linear order, as a modification of the oscillation matrices λ, but in the present case these contributions cancel when $-k$ is added to k.

For very strong coupling, the probability becomes simply periodic, as one mode would be in the absence of dissipation. This limit can be solved analytically, with the result that the oscillation frequency is the thermal average of the oscillation frequencies for the separate modes [10]. In fig 2 we show the entropy, as defined by $S = \sum_j \mathrm{Tr}[\rho(E_j)\log(\rho(E_j))]$, where the trace is over the 2×2 space. We note that as the coupling is increased from small to moderate values the rate of entropy increase rises, but that as the coupling becomes very large the rate of entropy increase goes to zero.

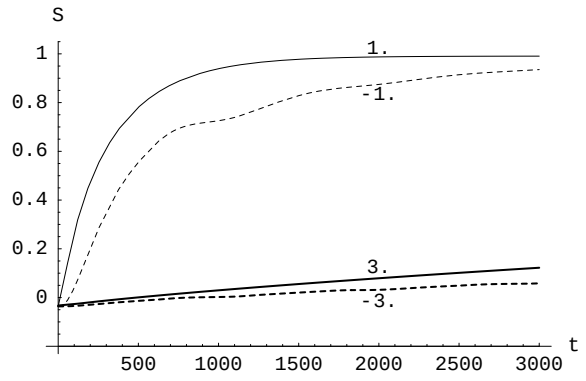

FIG. 2. The entropy S as a function of time for the same values of Log_{10}[coupling] used in fig.1.

We can generalize (3) to apply to a system in which the wells contain N fermions with no mutual interactions. For example, after making the appropriate modifications to the master equation for Fermi statistics, and using the same values of parameters as before, but taking $N = 5$, all initially on the left and in thermal equilibrium, we again look at the strong coupling limit, getting results very similar to those shown if fig.1 for the density matrix of a single particle. It is surprising that one can prepare a state in which, if we looked only at number density, a set of Fermi particles could appear to be oscillating, all in phase, between the left and the right, as would an assemblage of bosons all in an "average energy" state.

For the case of neutrino mixing in a noisy environment, where the vertical space is a continuum, the generalized Bloch equations, (3) are integro-differential equations,. But we find that when they are discretized with, say, around 30 energy levels, they are easily solved, with the results insensitive to the mesh size. The results are qualitatively the same as those discussed above.

There is an amusing generalization of the above considerations in the extension to a double well with a time dependent bias, $\delta H_0^{\rm sys} = \epsilon(t)\sigma_3$. Consider a case in which we begin with $\epsilon > 0$ and $\epsilon \gg |g(E_i)|$ for some set of low-lying states (but $\epsilon \ll \Delta E$). A group of particles initially localized on the left, distributed among the low-lying states, and not interacting with the medium, will stay largely on the left if the bias is maintained at the initial value. Now let the bias be slowly lowered until it is equal and opposite to the original bias. The particles will efficiently be carried to the right hand well. This is the analogue of the small angle MSW transition in neutrino physics. It is a rather diffuse transition, since the different energy levels go through their avoided crossings at different times. With a moderate coupling to the bath, the system instead settles down rather quickly to half on the right and half on the left, with no interesting persistent order. However, if we increase the bath coupling sufficiently we regain the shift from left to right. The transition is now sharper than it was in the case without the bath. Detailed analysis of this type of behavior for the neutrino case is given in [10].

Results for $b \neq 0$. We briefly summarize some examples. In the case plotted in fig. 1, for the strongest coupling, we find little change for $b = 0.001$. But $b = 0.005$ starts to poison the oscillation, damping it by a factor of 20% in 5 periods. As we move to much larger b, the motion becomes more frozen, so that when, e.g., $b = 0.5$, 98% of the probability has remained on the left hand side over a time span of five average periods of the $b = 0$ system. A popular interpretation of frozen behavior is that the interactions with the bath are continuously measuring the value of σ_3 and projecting the wave function onto the initial state. Although there is no measurement implied by our equations, the results are consistent with this terminology, since in the case with $b = 0$ the system-bath interaction does not distinguish the right-hand well from the left-hand well and it is for this case that we get the limiting behavior of undamped simple oscillations with a thermal averaged frequency.

Derivation of the master equation. We focus on the aspects that give rise to the placing of the horizontal matrices in (3) and the detailed form of the RHS of (5). The methods are closely related to those given in [11] and [12]. We introduce an interaction picture based on the above division of the Hamiltonian. The system-bath coupling in this picture is denoted by $H_I(t)$, with vertical matrix elements,

$$[H_I(t)]_{j,k} = e^{i\lambda(E_j)t}\zeta e^{-i\lambda(E_k)t}V^I_{j,k}e^{i(E_j-E_k)t}. \qquad (6)$$

The leading terms in the system+bath density matrix, $\rho_I^{\rm s+b}$, in this interaction picture are generated by the integral equation,

$$\rho_I^{\rm s+b}(t) = \rho^{\rm s+b}(t=0) - \int_0^t dt_1 \int_0^{t_1} dt_2 \Big[[\rho_I^{\rm s+b}(t_2), H_I(t_2)], H_I(t_1)\Big]. \quad (7)$$

In the the perturbation expansion of (7) we find a piece with an additional power of the quantities $c^2_{j,k}[Et, E\lambda^{-1}]$ in each higher order. This provides our definition of leading terms and the explanation for how a tiny $c^2_{j,k}$ can lead to appreciable effects over long periods of time. These terms are generated by the terms in which each successive new pair of H_I's in the iteration solution puts the state

back to the same vertical (and bath) state. These observations provide the basis for having omitted, in writing (7), all odd terms in H_I, and they also dictate the form of the density matrix to be used in solving the equation,

$$\rho_I^{\rm s+b} = \rho^{\rm bath} \sum_j \rho_I(E_j,t)|j\rangle\langle j|. \tag{8}$$

where,

$$\rho^{\rm bath} = \Big[\prod_i (1-e^{-\omega_i/T})\Big] \exp\Big[-(\sum_n a_n^\dagger a_n \omega_n)/T\Big], \tag{9}$$

and where,

$$\rho_I(E_j,t) = e^{i\lambda(E_j)t}\rho(E_j,t)e^{-i\lambda(E_j)t}. \tag{10}$$

We substitute (8) into (7), take the bath trace, and use leading order expressions for each term in the double commutator, obtaining, for example,

$$\begin{aligned}&{\rm Tr}_{\rm bath}\Big[\langle j|H_I(t_1)\rho_I^{\rm s+b}(t_2)H_I(t_2)|j\rangle\Big] = \delta(t_1-t_2)\\ &\times \sum_k \Gamma(E_k,E_j)e^{i\lambda(E_j)t_1}\zeta e^{-i\lambda(E_k)t_1}\\ &\times \rho_I(E_k,t_1)e^{i\lambda(E_k)t_1}\zeta e^{-i\lambda(E_j)t_1}.\end{aligned} \tag{11}$$

Similarly, we have

$$\begin{aligned}&{\rm Tr}_{\rm bath}\Big[\langle j|H_I(t_1)H_I(t_2)\rho_I^{\rm s+b}(t_2)|j\rangle\Big] = \delta(t_1-t_2)\\ &\times e^{i\lambda(E_j)t_1}\zeta^2 e^{-i\lambda(E_j)t_1}\sum_k \Gamma(E_j,E_k)\rho_I(E_j,t_1).\end{aligned} \tag{12}$$

The key to separation of the leading terms in the above is performing the sum over the modes of the scalar field first. After making the transition to the continuum, we make replacements of the form,

$$\begin{aligned}&\int_{\omega_{min}}^\infty d\omega f(\omega)e^{i(\pm E_j \mp E_k \pm\omega)(t_1-t_2)}\\ &\quad\to\quad 2\pi f[(E_k-E_j)]\delta(t_1-t_2)\theta(E_k-E_j).\end{aligned} \tag{13}$$

where neither the introduced infrared cut-off, ω_{min} nor the residual terms contribute to leading order, as defined above.

Using (11), (12) and their counterparts for the two other orderings in (6) , doing the t_2 integral (with the delta function symmetrically smeared, as can be justified by more accurate integrations) and differentiating with respect to t gives,

$$\begin{aligned}&\tfrac{d}{dt}\,\rho_I(E_j,t) = \sum_{E_k} e^{i\lambda(E_j)t}\zeta e^{-i\lambda(E_k)t}\rho_I(E_k,t)\\ &\times e^{i\lambda(E_k)t}\zeta e^{-i\lambda(E_j)t}\Gamma(E_k,E_j)\\ &-\frac{1}{2}\Big(e^{i\lambda(E_j)t}\zeta^2 e^{-i\lambda(E_j)t}\rho_I(E_j,t)\\ &+\rho_I(E_j,t)e^{i\lambda(E_j)t}\zeta^2 e^{-i\lambda(E_j)t}\Big)\sum_{E_k}\Gamma(E_j,E_k).\end{aligned} \tag{14}$$

Using (10) we now regain (3).

To summarize, we have developed the elements of the master equation for a bath-coupled system that is a generalization in two directions of the system of the ground states of a double-well. On the basis of this equation we have found new instances in which strong bath-system interactions can lead to more, rather than less, orderly behavior. We have not begun to address the question of identifying a realistic quantum well system in which one could make an experimental application of the conclusions. We should emphasize, however, that our limit of "large coupling" to the bath is the limit in which $\Gamma \gg \lambda$; it is achievable, in principle, for any strength of the system-bath coupling (whether the mechanism be photons, phonons or collisions with surrounding particles), simply by widening the barrier between the wells.

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